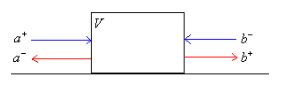
**1D Transfer Matrix**

**Transfer Matrix**

For repeated structures, there is a useful to device to handle extraction of the transmission/reflection coefficients. This involves the transfer matrix. We will assume the same asymptotic potentials on either end…and in this vein, suppose we specify the wavefunction ψL on the left, then ψR is related to ψL via a transfer matrix describing the interior of the sample. Set up looks like this:



and wavefunction looks like,



But the transfer matrix that we’ll be discussing relates the *current* amplitudes on the left side to those on the right side, and so it will be more convenient to put the wavefunction in terms of these amplitudes, rather than the particle beam density amplitudes. So we’ll write:



The a’s and b’s may be spinors if spin interactions are present. And matrix equation would look like this:



Let’s go ahead and work out an expression for this. We can write M in terms of the transmission and reflection matrices as well. Let’s demonstrate this. We’ll start with the relation involving the S-matrix.



Now rearrange to get the out/in coefficients:



Now invert and solve for the b’s,



To figure out the inverse of this matrix we write:



which results in:



So our inverse matrix is:



and we have:



And so we have:



The M11 term can apparently be simplified still. Consider:



Plugging in this expression



we get:



and so we see that we can write M as:



Just as a check, if we have (b+ 0) = M(a+ a­-), then we should get b+ = ta+, and a- = ra+. So…



So this works finally, after a million substitutions. Current conservation requires that current coming in, equals the current coming out, as usual. And the consequences for the transfer matrix are:



Flipping it around, we can write it as:



and is called the pseudo-unitarity condition. Filling in what M is…



and so we have as a consequence of flux conservation:



Note the ‘pseudo’ normalization condition of the two columns, and the ‘pseudo’ orthogonality condition. We can also show that M has unit determinant,

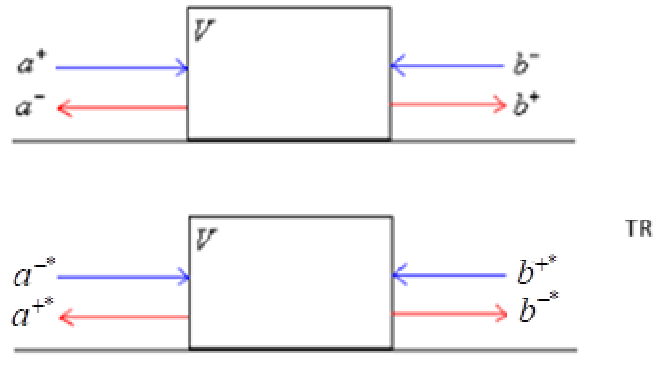


**Time Reversal Symmetry**

Time reversal symmetry says that ψ(x,-t)\* must be solution if ψ(x,t) is. Comparing the two wavefunctions:



Situation looks like this:



The transfer matrix for both would be:



for the same M. We can turn the last into the form of the first via:



which comes to the pseudo-symmetric condition:



In terms of the M matrix elements themselves, this relationship looks like:



So we have as a consequence of time reversal symmetry:

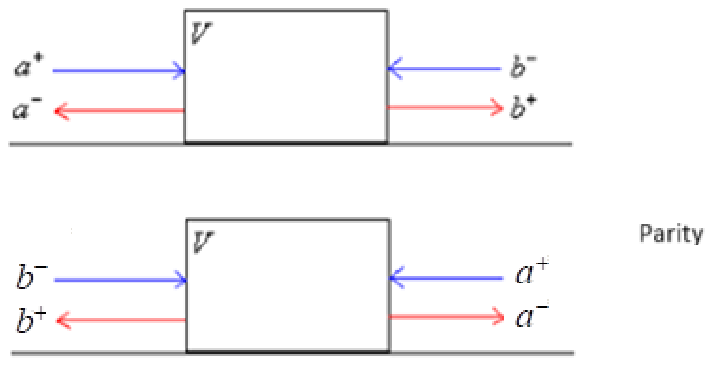


**Parity Symmetry**

What if we had parity symmetry too? Comparing the wavefunctions:



Situation looks like this:



So we have the two transfer matrix relations:



for the same M. The latter can be written in the form of the former via:



which implies



which would have the consequence:



Implications are:



So M is anti-symmetric and unimodular.

**Summary**

So altogether we have for M:

 (TRS, SRS are present): M is pseudo-unitary, and pseudo-orthogonal.

 (TRS is absent): M is pseudo-unitary.

 (TRS is present , SRS is absent): M is pseudo-unitary (and pseudo self-dual?)

In the β = 1 case, spin is just a spectator – there are no spin terms in H. Additionally, we have TRS, so the transmission coefficient is purely real – only 1 d.o.f. The β = 2 case refers to situations where we might have an L, or p term in H which doesn’t conserve TRS (but not a spin term, presumably). In this case the transmission matrix element can be complex – 2 d.o.f. In the last case, β = 4, we have to explicitly include spin states since we have spin-operators in the Hamiltonian. This gives us 4 d.o.f. in the transmission element since we can have spin up/down for incoming and spin up/down for outgoing.

**Change under translation**

Recalling what we said above,



M would change under translation as follows, under displacement Δx:



**Polar decomposition of M**

From our work on the S-matrix, we found that we could say:



Now let’s examine what M would look like in this polar decomposition. We have:



and this can be separated into:



If you change variables to Λ = R/T = (1-T)/T, then this can be written in a little nicer form



**Time Reversal Symmetry**

This requires that:



and comparing this to:



we see that:



And so with TRS we can write **M** as:



and this means that the transmission/reflection coefficients would be given by:



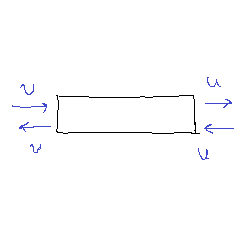
Another way to write this is as:



where u and υ are unitary matrices, and λ = (1-T)/T. The transmission/reflection coefficients are then given by:



Mello gives an informative interpretation of these relations. For instance, he says consider a unit current incident on a single channel, say n. Then the tmn matrix gives the amplitude of the current that is transmitted, while rmn gives the amplitude of reflection. Relating these quantities to the formulas above, he says that υ first mixes the current into the other channels, then 1/√(1+λ) and √λ/√(1+λ) determine the fraction in these channels that are transmitted/reflected respectively, and then υ, u mix the currents again before they leave the sample. Note u can be associated with right side, and υ with the left side of sample.



**Caveat for spin interactions**

If spin is important, like with SO interaction (which still preserves TRS because both L and S change sign), then we get the slightly weaker symmetry of ‘self-duality’. Parenthetically, having either/and B and SO involved is called a ‘spin-flip’ process. We have instead of:



rather



where the dual operation is defined as:



**Parity Symmetry**

Now let’s reexamine the consequences of parity symmetry, in the polar context. We earlier worked out that the equations obeyed were:



which, introducing the current conservation requirement,



allows us to write:



So there is just one independent equation, which is:



This would indicate that the product U\*V is purely imaginary. Okay, now let’s go back to the wavefunction. We have (bracketed expressions implicitly restricted to the left and right of the potential):



I’d like to see what it must look like asymptotically. We’ll recall that:



So we can at least say,



Now recall that,



so,



Now let’s say we’re dealing with the (+) parity eigenfunction. Then we must have ψ+(r) = ψ+(-r), which requires b- = a+ = aP+. And the odd (-) parity eigenfunction would require a+ = -b- = -bP-, where we define new arbitrary constants indicative of their parity value. So we may write:



Let’s work these out a little more,



The quantities t + r and t – r are pure phases, as one may verify,



and,



So let’s call



and then we can write the two wavefunctions as:



which is, ignoring the middle part,



If we factor out one of the phase factors then we have:



This is precisely the form we found when examining the free states of some of those 1D potentials. We may work out t and r themselves in terms of the phases,



I think this would be the analogue to the 3D equations relating the T-matrix and phase shift.

We can do a little more work on t and r to put them in terms of their magnitude and phase. Consider factoring out the magnitude of t from t, to get the phase,



and let’s do the same with r,



And so we have:



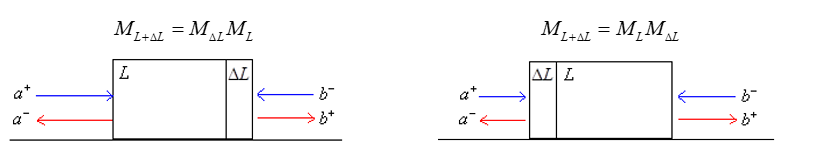
So we do need both phase shifts to get either t or r it seems. The transmission and reflection coefficients follow from this, via T = |t|2 and R = |r|2,



We can clearly see that T + R = 1 from this expression. Another thing we can see is that when either δ+ or δ- hit π/2, then we tend to have a maximimum in R and a minimum in T. Conversely, if one of them goes to 0, then we tend to have a minimum a maximum in T and minimum in R. This situation will play out in 3D too.

**Convolution Property of M**

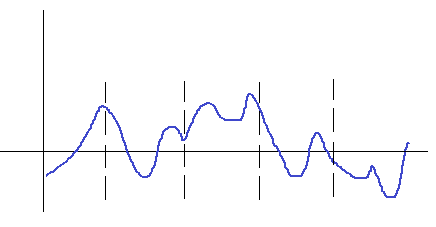
The advantage of working with these matrices is that M obeys the multiplicative properties shown below.



But technically, one would have to use the infinite dimensional version, which includes all open and closed channels,



because in say, the potential below, there is nothing that prevents exponentials within the finite ΔL regions.



This is even so if there are free regions on both sides of the potential within that ΔL region. Still, in the limit of large open channels, Mello says that we may often neglect these evanescent modes, especially if in the S-matrix setup, the evanescent modes associated with the N open incoming/outgoing channels tend to exponentially decay on a length scale smaller than ΔL.